

A CLT FOR WEIGHTED TIME-DEPENDENT UNIFORM EMPIRICAL PROCESSES

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ABSTRACT. For a uniform process $\{X_t : t \in E\}$ (by which X_t is uniformly distributed on $(0, 1)$ for $t \in E$) and a function $w(x) > 0$ on $(0, 1)$, we give a sufficient condition for the weak convergence of the empirical process based on $\{w(x)(1_{X_t \leq x} - x) : t \in E, x \in [0, 1]\}$ in $\ell^\infty(E \times [0, 1])$. When specializing to $w(x) \equiv 1$ and assuming strict monotonicity on the marginal distribution functions of the input process, we recover a result of [9]. In the last section, we give an example of the main theorem.

1. INTRODUCTION

Given a sequence of independent uniform $(0, 1)$ random variables X_1, X_2, \dots , if let $G_n(x) = n^{-1/2} \sum_{i=1}^n (1_{X_i \leq x} - x)$ be the uniform empirical process, then Donsker's theorem ([4]) says $G_n(x)$ converges weakly to the Brownian bridge process, $B(x)$, on $[0, 1]$. Weighted empirical processes consider suitable weight functions $w(x)$ such that $w(x)G_n(x)$ converges weakly to the weighted Brownian bridge process $w(x)B(x)$; in the literature, such a theorem is called Chibisov-O'Reilly theorem; see [2], [12], [3] etc. [9] considered a time dependent empirical process

$$G_n(t, y) := n^{-1/2} \sum_{i=1}^n (1_{Y_i(t) \leq y} - P(Y_i(t) \leq y)), \quad t \in E, \quad y \in \mathbb{R},$$

for independent and identically distributed (iid) stochastic processes $Y_1(t), Y_2(t), \dots$ for $t \in E$. Under a condition the authors call the L-condition, this empirical process converges weakly in $\ell^\infty(E \times \mathbb{R})$. In [5], the authors proved a CLT for weighted tail empirical processes under a small oscillation condition as the L-condition guarantees.

We consider a time dependent weighted uniform empirical process. For a process $X(t)$ for $t \in E$ and a "weight function" $w(x)$ on $(0, 1)$, we are interested in conditions on the process and the weight function so that the empirical process

$$\nu_n(t, y) := n^{-1/2} \sum_{i=1}^n w(y)(1_{X_i(t) \leq y} - y), \quad t \in E, \quad y \in [0, 1],$$

where $X(t), X_1(t), X_2(t), \dots$ are iid, converges weakly in $\ell^\infty(E \times [0, 1])$. We give a sufficient condition in Section 3 for a Central limit theorem (CLT) for this empirical process.

This paper is organized as following. In Section 2, we give some definitions and results about weak convergence (CLT) for empirical processes. Section 3 contains

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the main result. The proof is to use Theorem 4.4 in [1]. In particular, the pre-Gaussian condition and the local modulus condition are to be checked under the assumptions. An example of the main theorem is given at the last section.

2. PRELIMINARIES

Given a centered stochastic process $\{X(t) : t \in T\}$, we define the empirical process based on it by

$$(2.1) \quad \nu_n(t) := n^{-1/2} \sum_{j=1}^n X_j(t), \quad t \in T,$$

where $\{X_j(t) : t \in T\}$ for $j = 1, 2, \dots$ are independent and identically distributed as $\{X(t) : t \in T\}$.

On a probability space (Ω, \mathcal{A}, P) , recall the outer expectation of an arbitrary function $f : \Omega \rightarrow \mathbb{R}$

$$E^*(f) := \inf\{Eg : g \geq f, g \text{ is } (\mathcal{A}, \mathcal{B}(\mathbb{R})) \text{ measurable}\}.$$

Definition 2.1. Let $X := \{X(t) : t \in T\}$ be a centered stochastic process on a parameter set T , and sample paths in $\ell^\infty(T)$. Assume $E|X(t)|^2 < \infty$ for $t \in T$. The empirical process based on X , $\nu_n(t)$ in (2.1), satisfies the central limit theorem, – for short $X \in \text{CLT}$ – if there exists a centered Radon measure γ on $\ell^\infty(T)$ such that for all $H : \ell^\infty(T) \rightarrow \mathbb{R}$ bounded and continuous, we have

$$\lim_{n \rightarrow \infty} E^*(H(\nu_n)) = \int_{\ell^\infty(T)} H d\gamma.$$

Definition 2.2. A centered stochastic process $\{X_t : t \in T\}$ is pregaussian if its covariance coincides with the covariance of a centered Gaussian process G on T with bounded and uniformly d_G -continuous sample paths, where $d_G(s, t) := (E(G(s) - G(t))^2)^{1/2}$.

Theorem 2.3 (cf. [9], Proposition 1). *Let H_1 and H_2 be zero mean Gaussian processes with L_2 distances d_{H_1} , d_{H_2} , respectively, on T . Furthermore, assume T is countable, and $d_{H_1}(s, t) \leq d_{H_2}(s, t)$ for all $s, t \in T$. Then, H_2 sample bounded and uniformly continuous on (T, d_{H_2}) with probability one, implies H_1 is sample bounded and uniformly continuous on (T, d_{H_1}) with probability one.*

When $T = [0, 1]$, this is Lemma 2.1 in [11].

The assumption that T is countable can be removed if T is given a totally bounded metric.

Lemma 2.4. *Let $\{G(t) : t \in T\}$ be a zero mean Gaussian process. Further assume $\sup_{t \in T} EG(t)^2 < \infty$. Let $d_G(s, t) := (E(G(s) - G(t))^2)^{1/2}$. Then, if T_0 is a dense set in (T, d_G) and the restricted process $\{G(t) : t \in T_0\}$ is sample bounded and uniformly d_G -continuous, then $\{G(t) : t \in T\}$ has a version with bounded and uniformly d_G -continuous sample paths.*

The proof of this lemma is given in the appendix.

We will use the following theorem to prove our main result.

Theorem 2.5 ([1], Theorem 4.4). *Let $\{X(t) : t \in T\}$ be a sample bounded process on a set T such that $EX(t) = 0$ and $EX(t)^2 < \infty$ for all $t \in T$. Assume:*

- (i) $u^2 P^*\{\|X\|_\infty > u\} \rightarrow 0$ as $u \rightarrow \infty$,

- (ii) X is pregaussian, and
- (iii) there is pseudometric ρ on T dominated by the pseudometric d_G corresponding to a centered Gaussian process G on T with bounded and uniformly d_G -continuous paths such that for some K and for all $t \in T$ and $\varepsilon > 0$,

$$\sup_{u>0} u^2 P^* \left(\sup_{s \in B_\rho(t, \varepsilon)} |X(t) - X(s)| > u \right) \leq K \varepsilon^2.$$

Then $X \in \text{CLT}$ as a $\ell^\infty(T)$ -valued random element.

Definition 2.6. Let $F(x)$ be a distribution function (df) on \mathbb{R} . The (randomized) distributional transform of $F(x)$ as defined in [13] is

$$\tilde{F}(x) := \tilde{F}(x, V) := F(x-) + (F(x) - F(x-))V,$$

where V is a uniform random variable on $[0, 1]$.

Next we give some simple properties of the distributional transform.

- Lemma 2.7.** (i) $\tilde{F}(x) \leq F(x)$ for all $x \in \mathbb{R}$.
(ii) If $x < y$, then $F(x) \leq \tilde{F}(y)$.
(iii) If $x \leq y$, then $\tilde{F}(x) \leq \tilde{F}(y)$.
(iv) If $x < y$ and $F(\cdot)$ is strictly increasing, then $F(x) < \tilde{F}(y)$.

Proof. By definition, (i) is obvious. For (ii), take $x < z < y$, hence $F(x) \leq F(z)$. Since $F(z) \leq F(y-)$ and $F(y-) \leq \tilde{F}(y)$, hence $F(x) \leq \tilde{F}(y)$. For (iii), if $x = y$, there is nothing to prove; assume $x < y$. By (i) and (ii), we get (iii). For (iv), take $x < z < y$. Since $F(\cdot)$ is strictly increasing, $F(x) < F(z)$. But by (ii), $F(z) \leq \tilde{F}(y)$. Hence $F(x) < \tilde{F}(y)$. \square

For a continuous df F of a random variable X , the random variable $F(X)$ is uniform on $[0, 1]$; but for a general df F , this might not be the case. However using the (randomized) distributional transform overcomes this.

Lemma 2.8. If $F(x)$ is the distribution function of a random variable X , then $\tilde{F}(X) := \tilde{F}(X, V)$ is uniform on $[0, 1]$. Here V is a uniform random variable on $[0, 1]$ independent of X .

Proof. For a proof, see [13]. \square

Definition 2.9. We say a (pseudo) distance ρ on a set T is a continuous Gaussian distance if there is a zero mean Gaussian process $\{G(t) : t \in T\}$ with bounded and uniformly d_G -continuous sample paths where $d_G(s, t) := (\mathbb{E}(G(s) - G(t))^2)^{1/2}$ and $\rho(s, t) = d_G(s, t)$ for all $s, t \in T$.

For notation, we write X for a process $\{X(t) : t \in E\}$ and X_t for $X(t)$. We recall from [9]

Definition 2.10 (L-condition for a stochastic process). Let $X := \{X_t : t \in E\}$ be a stochastic process. The process X satisfies the L-condition if there exists a continuous Gaussian distance ρ on E such that for every $\varepsilon > 0$

$$(2.2) \quad \sup_{t \in E} P^* \left(\sup_{s: \rho(s, t) \leq \varepsilon} |\tilde{F}_t(X_t) - \tilde{F}_t(X_s)| > \varepsilon^2 \right) \leq L \varepsilon^2,$$

where $\tilde{F}_t(\cdot)$ is the distributional transform of the distribution function $F_t(\cdot)$ of X_t .

Theorem 2.11 ([9], Theorem 3). *Let $X(t)$ be a process on E . Let ρ be given by $\rho(s, t)^2 = E(H(s) - H(t))^2$, for some centered Gaussian process H that is sample bounded and uniformly continuous on (E, ρ) with probability one. Further, assume that for some $L < \infty$, and all $\varepsilon > 0$, the L -condition holds for X , and $D(E)$ is a collection of real valued functions on E such that $P(X(\cdot) \in D(E)) = 1$. If*

$$\mathcal{C} = \{C_{s,x} : s \in E, x \in \mathbb{R}\},$$

where

$$C_{s,x} = \{z \in D(E) : z(s) \leq x\}$$

for $s \in E$, $x \in \mathbb{R}$, then $\mathcal{C} \in \text{CLT}(X)$.

In this case, we say the empirical process based on $\{1_{Y(t) \leq y} - P(Y(t) \leq y) : t \in E, y \in \mathbb{R}\}$ satisfies the CLT or write $\mathcal{C} \in \text{CLT}(X)$ in $\ell^\infty(E \times \mathbb{R})$.

3. WEAK CONVERGENCE OF THE TIME DEPENDENT WEIGHTED EMPIRICAL PROCESS

In view of Theorem 2.11 and the classical weighted empirical process, a natural question is to consider the time dependent weighted (uniform) empirical process,

$$\alpha_n(t, y) := n^{-1/2} \sum_{i \leq n} w(y)(1_{X_i(t) \leq y} - y), t \in E, y \in [0, 1]$$

where $\{X(t), X_1(t), X_2(t), \dots\}$ are iid uniform processes (see the definition below). Under the WL-condition (below) and some regularity conditions on the weight function $w(\cdot)$, we prove a CLT for the empirical process α_n .

Definition 3.1. We call a process $X = \{X(t) : t \in E\}$ a uniform process if for each $t \in E$, $X(t)$ is uniformly distributed on $(0, 1)$.

We call the main condition in our theorem the WL-condition.

Definition 3.2. [WL-condition for $(X; w)$] Given a uniform process $X := \{X_t : t \in E\}$ and a function $w := w(x) > 0$ on $(0, 1)$, we say $(X; w)$ satisfies the WL-condition if for some constant L (depending on w , but not on x), some continuous Gaussian distance ρ on E and all $\varepsilon > 0$, $0 < x < 1$, we have

$$(3.1) \quad \sup_t P^* \left(\sup_{s: \rho(s, t) \leq \varepsilon} 1_{X_t \leq x < X_s} > 0 \right) \leq \frac{L\varepsilon^2}{w(x)^2}$$

$$(3.2) \quad \sup_t P^* \left(\sup_{s: \rho(s, t) \leq \varepsilon} 1_{X_s \leq x < X_t} > 0 \right) \leq \frac{L\varepsilon^2}{w(x)^2}$$

The following is the main result of this paper.

Theorem 3.3. *Let $X := \{X_t : t \in E\}$ be a uniform process on a parameter set E . Let $w := w(x) > 0, 0 < x < 1$ be continuous and symmetric about $x = 1/2$ for which there exists $\gamma \in (0, 1/2]$ such that w is non-increasing and $xw(x)^2$ is non-decreasing on $(0, \gamma)$ and such that w is uniformly bounded on $[\gamma, 1/2]$. Further, assume that $w(x)$ is regularly varying in a neighborhood of zero and satisfies the integral condition*

$$(3.3) \quad \int_0^\gamma s^{-1} \exp[-c/(sw(s)^2)] ds < \infty \text{ for all } c > 0.$$

If

$$\lim_{\alpha \rightarrow \infty} \alpha^2 P^*(\sup_{t \in E} w(X_t) > \alpha) = 0$$

and the WL-condition for $(X; w)$ is satisfied, then the empirical process based on $\{w(x)(1_{X_t \leq x} - x) : t \in E, x \in [0, 1]\}$ converges weakly in $\ell^\infty(E \times [0, 1])$.

Remark 3.4. (1) We require that the function $w(x)$ be symmetric about $1/2$ is no loss of generality. As the Brownian bridge has the same behavior at 0 and 1. Moreover we only give the proof of the theorem for $0 < x < 1/2$. Indeed, if let $\tilde{X}_t := 1 - X_t$, then $(\tilde{X}; w)$ satisfies the WL-condition. The result for \tilde{X} for $0 < x \leq 1/2$ gives a result of X for $1/2 < x \leq 1$. The fact (cf. [8], Corollary 1.6, p. 61) that if \mathcal{F}_1 and \mathcal{F}_2 are Donsker classes, then $\mathcal{F} := \mathcal{F}_1 \cup \mathcal{F}_2$ is a Donsker class gives the result for $\mathcal{F} = E \times [0, 1]$.

(2) For a general process $Y := \{Y_t : t \in E\}$, if we define $X := X_t := \tilde{F}_t(Y_t)$, where $\tilde{F}_t(\cdot)$ is the (randomized) distributional transform of the df F_t of Y_t , then X is a uniform process (see Lemma 2.8). Such a process X is called a copula process. If we have a CLT for the X process, then we have a CLT for the Y process; see Proposition 3.5 for precise statement. In case of $w \equiv 1$, this theorem gives a proof of Theorem 2.11 provided that $F_t(\cdot)$ for each $t \in E$ is strictly increasing; see Corollary 3.6.

(3) The integral condition (3.3) is necessary and sufficient for one dimensional weighted uniform empirical process under regularity of the weight function; see [1], Example 4.9.

The proof of the theorem is given at the end of this section.

The following is a possible way that a CLT for the time dependent empirical process for Y can be obtained from proving a CLT for the process X .

Proposition 3.5. *Let $w(x)$ be any function on $(0, 1)$. Let $\{Y_t : t \in E\}$ be a process and $F_t(\cdot)$ is the df of Y_t . Let $X_t := \tilde{F}_t(Y_t)$. Then the following hold:*

(i) *If $F_t(\cdot)$ is strictly increasing for each $t \in E$, then*

$$\{w(x)(1_{X_t \leq x} - x) : t \in E, x \in [0, 1]\} \in \text{CLT in } \ell^\infty(E \times [0, 1])$$

implies

$$\{w(F_t(y))(1_{Y_t \leq y} - F_t(y)) : t \in E, y \in \mathbb{R}\} \in \text{CLT in } \ell^\infty(E \times \mathbb{R}).$$

(ii) *Without assuming that $F_t(\cdot)$ is strictly increasing for each $t \in E$, we have*

$$\{w(x)(1_{X_t \leq x} - x) : t \in E, x \in [0, 1]\} \in \text{CLT in } \ell^\infty(E \times [0, 1])$$

implies

$$\{w(F_t(y))(1_{Y_t \leq y} - F_t(y)) : (t, y) \in T_0\} \in \text{CLT in } \ell^\infty(T_0),$$

where T_0 is any countable subset of $E \times \mathbb{R}$.

Proof. Proof of (i). Recall that $\tilde{F}(x) \leq \tilde{F}(y)$ for $x \leq y$ and $\tilde{F}(x) \leq F(x)$ for all $x \in \mathbb{R}$ and for any df F (see Lemma 2.7). Hence $Y_t \leq y$ implies that $\tilde{F}_t(Y_t) \leq F_t(y)$; i.e.

$$(3.4) \quad 1_{Y_t \leq y} \leq 1_{\tilde{F}_t(Y_t) \leq F_t(y)}, \text{ uniformly in } t \in E, y \in \mathbb{R}.$$

Since $F_t(\cdot)$ is strictly increasing, by the same lemma if $x < y$, then $F(x) < \tilde{F}(y)$. Now if $\tilde{F}_t(Y_t) \leq F_t(y)$ and $Y_t > y$ for some $t \in E$ and $y \in \mathbb{R}$, then $F_t(y) < \tilde{F}_t(Y_t)$. We have a contradiction: $F_t(y) < F_t(y)$. Thus $\tilde{F}_t(Y_t) \leq F_t(y)$ implies $Y_t \leq y$; i.e.

$$1_{Y_t \leq y} \geq 1_{\tilde{F}_t(Y_t) \leq F_t(y)}, \text{ uniformly in } t \in E, y \in \mathbb{R}.$$

Combining the two displays, we have

$$(3.5) \quad 1_{Y_t \leq y} = 1_{\tilde{F}_t(Y_t) \leq F_t(y)}, \text{ uniformly in } t \in E, y \in \mathbb{R}.$$

Since $\{F_t(y) : t \in E, y \in \mathbb{R}\}$ is a subset of $[0, 1]$, thus if the empirical process based on $\{w(x)(1_{\tilde{F}_t(Y_t) \leq x} - x) : t \in E, x \in [0, 1]\}$ satisfies CLT in $\ell^\infty(E \times [0, 1])$, then, by substituting x with $F_t(y)$ and using (3.5), the empirical process based on $\{w(F_t(y))(1_{Y_t \leq y} - F_t(y)) : t \in E, y \in \mathbb{R}\}$ satisfies the CLT in $\ell^\infty(E \times \mathbb{R})$.

Proof of (ii). Fix $t \in E$ and $y \in \mathbb{R}$. If $\tilde{F}_t(Y_t) \leq F_t(y)$, since $\tilde{F}_t(Y_t) = F_t(y)$ has probability zero, then, after throwing out this null set, $\tilde{F}_t(Y_t) < F_t(y)$, which will imply $Y_t \leq y$. If not, then $Y_t > y$, by Lemma 2.7, hence $F_t(y) \leq \tilde{F}_t(Y_t)$. Again we have a contradiction $F_t(y) < F_t(y)$. Thus almost surely $1_{\tilde{F}_t(Y_t) \leq F_t(y)} \leq 1_{Y_t \leq y}$. Combining this with 3.4 gives, almost surely,

$$(3.6) \quad 1_{Y_t \leq y} = 1_{\tilde{F}_t(Y_t) \leq F_t(y)}, \text{ uniformly in } (t, y) \in T_0,$$

where T_0 is any countable set in $E \times \mathbb{R}$. Restricting to the countable set, we have the stated implication as in (i). \square

Corollary 3.6 (cf. [9], Theorem 3). *Let $Y := \{Y_t : t \in E\}$ be a process. Let F_t be the df of Y_t . In addition, assume that $F_t(\cdot)$ is strictly increasing for each $t \in E$ and that Y satisfies the L-condition:*

$$(3.7) \quad \sup_{t \in E} \mathbb{P}^* \left(\sup_{s: \rho(s, t) \leq \varepsilon} |\tilde{F}_t(Y_t) - \tilde{F}_t(Y_s)| > \varepsilon^2 \right) \leq L\varepsilon^2,$$

for a constant L and a continuous Gaussian metric $\rho(s, t)$ on E . Then

$$\{1_{Y_t \leq y} - \mathbb{P}(Y_t \leq y) : t \in E, y \in \mathbb{R}\} \in \text{CLT in } \ell^\infty(E \times \mathbb{R}).$$

Remark 3.7. Under the L-condition, we will see from the proof of Theorem 3.12 that there is a countable dense set in $E \times \mathbb{R}$ with respect to the L_2 distance of the limiting Gaussian process. Hence without the restriction that $F_t(\cdot)$ is strictly increasing, we still have a CLT but on a countable dense set.

Proof of Corollary 3.6. By part (i) of Proposition 3.5, we only need to check the conditions in Theorem 3.3 with $w(x) \equiv 1$.

Under the L-condition, we have (cf. [9], Lemma 1)

$$\sup_x |F_t(x) - F_s(x)| \leq 2(L+1)\rho(s, t)^2.$$

Consequently by passing to the limit,

$$\sup_x |F_t(x-) - F_s(x-)| \leq 2(L+1)\rho(s, t)^2.$$

Recalling that $\tilde{F}_s(x) = F_s(x-) + V(F_s(x) - F_s(x-))$, we obtain

$$\begin{aligned} \sup_x |\tilde{F}_t(x) - \tilde{F}_s(x)| &\leq \sup_x |F_t(x-) - F_s(x-)| + \sup_x |V(F_t(x) - F_s(x))| \\ &\quad + \sup_x |V(F_t(x-) - F_s(x-))| \\ &\leq 6(L+1)\rho(s, t)^2. \end{aligned}$$

For $t \in E$ fixed, let $A := \{ \sup_{s: \rho(s,t) \leq \varepsilon} |\tilde{F}_t(Y_t) - \tilde{F}_t(Y_s)| > \varepsilon^2 \}$.

On the complement, A^c , of A , we have for all s with $\rho(s,t) \leq \varepsilon$,

$$\begin{aligned} |\tilde{F}_s(Y_s) - \tilde{F}_t(Y_t)| &\leq |\tilde{F}_s(Y_s) - \tilde{F}_t(Y_s)| + |\tilde{F}_t(Y_s) - \tilde{F}_t(Y_t)| \\ &\leq 6(L+1)\rho(s,t)^2 + \varepsilon^2 \\ &\leq (6L+7)\varepsilon^2. \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{P}^* \left(\sup_{s: \rho(s,t) \leq \varepsilon} 1_{\tilde{F}_s(Y_s) \leq x < \tilde{F}_t(Y_t)} > 0 \right) &= \mathbf{P}^*(A^c, \sup_{s: \rho(s,t) \leq \varepsilon} 1_{\tilde{F}_s(Y_s) \leq x < \tilde{F}_t(Y_t)} > 0) \\ &\quad + \mathbf{P}^*(A, \sup_{s: \rho(s,t) \leq \varepsilon} 1_{\tilde{F}_s(Y_s) \leq x < \tilde{F}_t(Y_t)} > 0) \\ &\leq \mathbf{P}(A^c, 1_{\tilde{F}_t(Y_t) - (6(L+1)\varepsilon^2 + \varepsilon^2) \leq x < \tilde{F}_t(Y_t)} > 0) + L\varepsilon^2 \end{aligned}$$

Keeping in mind that $\tilde{F}_t(Y_t) \stackrel{d}{=} U(0,1)$

$$\leq (7L+7)\varepsilon^2.$$

Similarly,

$$\mathbf{P}^* \left(\sup_{s: \rho(s,t) \leq \varepsilon} 1_{\tilde{F}_t(Y_t) \leq x < \tilde{F}_s(Y_s)} > 0 \right) \leq (7L+7)\varepsilon^2.$$

In addition, obviously for $w(x) \equiv 1$

$$\lim_{\alpha \rightarrow \infty} \alpha^2 \mathbf{P}^* \left(\sup_{t \in E} w(\tilde{F}_t(Y_t)) > \alpha \right) = 0.$$

Thus we have verified the conditions in Theorem 3.3. \square

We will prove Theorem 3.3 only for $0 < x < 1/2$ as explained in Remark 3.4. We will check the pre-Gaussian condition (ii) and the local modulus condition (iii) in Theorem 2.5.

3.1. Pre-Gaussian. Let $\{G_0((s,x)) : s \in E, x \in [0,1]\}$ be the zero mean Gaussian process with covariance

$$(3.8) \quad EG_0(s,x)G_0(t,y) = w(x)w(y)\mathbf{P}(X_s \leq x, X_t \leq y).$$

Under the assumptions of Theorem 3.3, we will prove $G_0(s,x)$ has a version with bounded and uniformly continuous sample paths with its L_2 distance d_{G_0} by comparing it with some other continuous Gaussian distance; consequently by another comparison the centered Gaussian process with covariance

$$(3.9) \quad EG(s,x)G(t,y) := w(x)w(y)[\mathbf{P}(X_s \leq x, X_t \leq y) - xy]$$

has a version with bounded and uniformly continuous sample paths with its L_2 distance d_G , which is equivalent to say the process $\{w(y)(1_{X_t \leq y} - y) : t \in E, y \in [0,1]\}$ is pre-Gaussian.

Lemma 3.8 (see [1], Example 4.8). *Let $W(y)$ be a Brownian motion and $w(y)$ as in Theorem 3.3. Then the Gaussian process $\{w(y)W(y) : y \in [0,1]\}$ is sample bounded and uniformly continuous w.r.t. its L_2 distance, which is given by*

$$(3.10) \quad d(x,y)^2 := \mathbf{E}(w(y)W(y) - w(x)W(x))^2 = w(x \vee y)^2|y-x| + (x \wedge y)(w(x) - w(y))^2.$$

Lemma 3.9. *If $xw(x)^2$ is non-decreasing and $w(x)$ is non-increasing for $0 < x < \delta$, then*

$$d(x, y) \leq d(x, z)$$

for $0 < x \leq y \leq z \leq \delta$.

Proof. Let $0 < x \leq y \leq z \leq \delta$. Using definition (3.10) and the monotonicity of $xw(x)^2$ and $w(x)$, we obtain

$$\begin{aligned} d(x, y)^2 &= w(y)^2(y - x) + x(w(y) - w(x))^2 \\ &= xw(x)^2 + yw(y)^2 - 2xw(x)w(y) \\ &\leq xw(x)^2 + zw(z)^2 - 2xw(x)w(z) \\ &= d(x, z)^2. \end{aligned}$$

□

Next we give an upper bound for d_{G_0} under WL-condition in Theorem 3.3.

Lemma 3.10. *Let $d(x, y)$ be as in (3.10) and $d_{G_0}((s, x), (t, y))$ the L_2 distance of the Gaussian process G_0 in (3.8). Then under the WL-condition, we have*

$$d_{G_0}^2((s, x), (t, y)) \leq 2d^2(x, y) + 4L\rho(s, t)^2.$$

Proof. First observe that for $t \in E$

$$(3.11) \quad d(x, y)^2 = E(w(y)W(y) - w(x)W(x))^2 = E|w(x)1_{X_t \leq x} - w(y)1_{X_t \leq y}|^2.$$

Using, by the WL-condition for fixed s and t ,

$$(3.12) \quad P(X_s \leq x < X_t) \leq \frac{L\rho(s, t)^2}{w(x)^2} \text{ and } P(X_t \leq x < X_s) \leq \frac{L\rho(s, t)^2}{w(x)^2},$$

we obtain

$$\begin{aligned} (3.13) \quad d_{G_0}^2((s, x), (t, y)) &= E|w(x)1_{X_s \leq x} - w(y)1_{X_t \leq y}|^2 \\ &= E|w(x)1_{X_s \leq x} - w(x)1_{X_t \leq x} + w(x)1_{X_t \leq x} - w(y)1_{X_t \leq y}|^2 \\ &\leq 2E|w(x)1_{X_s \leq x} - w(x)1_{X_t \leq x}|^2 + 2E|w(x)1_{X_t \leq x} - w(y)1_{X_t \leq y}|^2 \\ &= 2w(x)^2E|1_{X_s \leq x} - 1_{X_t \leq x}|^2 + 2d(x, y)^2 \quad \text{by (3.11)} \\ &\leq 2w(x)^2(P(X_s \leq x < X_t) + P(X_t \leq x < X_s)) + 2d(x, y)^2 \\ &\leq 4L\rho(s, t)^2 + 2d(x, y)^2 \quad \text{by (3.12).} \end{aligned}$$

□

Corollary 3.11. *Under the WL-condition, the process $G_0(t, y)$ is sample bounded and uniformly continuous with respect to its L_2 distance; the same is true for a zero mean Gaussian process with covariance*

$$(3.14) \quad EG(s, x)G(t, y) := w(x)w(y)[P(X_s \leq x, X_t \leq y) - xy].$$

Proof. By assumption, ρ is the L_2 distance of a zero mean Gaussian process on E , say $\{H_0(t) : t \in E\}$, with bounded and uniformly ρ -continuous sample paths. Let the metric d on $[0, 1]$ as given in 3.10 with the corresponding Gaussian process $w(x)W(x)$, which is sample bounded and uniformly d -continuous. Let $H_2((t, y)) := 2^{1/2}w(y)W(y) + 2L^{1/2}H_0(t) : t \in E, y \in [0, 1]$, where W and H_0 are independent. Then the L_2 distance, $d_{H_2}((s, x), (t, y))$, of H_2 is $2^{1/2}d(x, y) + 2L^{1/2}\rho(s, t)$. Total boundedness of d and ρ implies that of d_{H_2} . Thus let T_0 be a dense subset in $(E \times [0, 1], d_{H_2})$; since $d_{G_0} \leq d_{H_2}$ by (3.13), T_0 is also a dense subset in $(E \times$

$[0, 1], d_{G_0}$; Using the comparison Theorem 2.3 with $H_1 := G_0$ and that $d_{G_0} \leq d_{H_2}$, the Gaussian process $\{G_0 : (s, x) \in T_0\}$ is sample bounded and uniformly d_{G_0} continuous. By Lemma 2.4, $\{G_0 : (s, x) \in E \times R\}$ is sample bounded and uniformly d_{G_0} -continuous; the second statement in the Lemma is straightforward. \square

For the pre-Gaussian property of the empirical process considered in [9], we give a different proof rather than the constructive one in [9] using the generic chaining [15].

Theorem 3.12. *Let $\{Y(t) : t \in E\}$ be a process and satisfies the L-condition, then the centered Gaussian process on $E \times \mathbb{R}$ with covariance either*

$$P(Y_s \leq x, Y_t \leq y) - P(Y_s \leq x)P(Y_t \leq y)$$

or

$$P(Y_s \leq x, Y_t \leq y)$$

has a version, which is sample bounded and uniformly continuous with respect to its L_2 distance.

Proof. Let $\{G_1(t, y) : t \in E, y \in \mathbb{R}\}$ and $\{G_2(t, y) : t \in E, y \in \mathbb{R}\}$ be the Gaussian processes on $E \times \mathbb{R}$ with covariance $P(Y_s \leq x, Y_t \leq y) - P(Y_s \leq x)P(Y_t \leq y)$ and $P(Y_s \leq x, Y_t \leq y)$, respectively. Let d_{G_1} and d_{G_2} be their L_2 distances, respectively; i.e.,

$$(3.15) \quad \begin{aligned} d_{G_1}((s, x), (t, y))^2 &= E(1_{Y_s \leq x} - 1_{Y_t \leq y})^2 - (E(1_{Y_s \leq x} - 1_{Y_t \leq y}))^2, \\ d_{G_2}((s, x), (t, y))^2 &= E(1_{Y_s \leq x} - 1_{Y_t \leq y})^2. \end{aligned}$$

And,

$$(3.16) \quad \begin{aligned} d_{G_2}((s, x), (t, y))^2 &= E(1_{Y_s \leq x} - 1_{Y_t \leq y})^2 \\ &= E(1_{Y_s \leq x} - 1_{Y_t \leq x} + 1_{Y_t \leq x} - 1_{Y_t \leq y})^2 \\ &\leq 2E(1_{Y_s \leq x} - 1_{Y_t \leq x})^2 + E(1_{Y_t \leq x} - 1_{Y_t \leq y})^2 \\ &\leq 2(P(Y_s \leq x < Y_t) + P(Y_t \leq x < Y_s)) + |F_t(y) - F_t(x)| \\ &\leq 6(L + 1)\rho(s, t)^2 + |F_t(y) - F_s(x)|, \end{aligned}$$

where in the last line of the above display, we used Lemma 1 in [9].

Let $W(\cdot)$ be a Brownian motion on $[0, \infty)$. Define the centered Gaussian process

$$H_2(t, y) := W(F_t(y)) : t \in E, y \in \mathbb{R},$$

where $F_t(\cdot)$ be the df of Y_t . Then its L_2 distance $d_{H_2}((s, x), (t, y)) = |F_t(y) - F_s(x)|^{1/2}$. By the uniform continuity of the sample paths of $W(\cdot)$ on $[0, 1]$, it follows that H_2 is sample bounded and uniformly continuous with respect to d_{H_2} . By the L-condition, let $\{H_1(t) : t \in E\}$, independent from H_2 , be a Gaussian process with bounded and uniformly continuous sample paths with its L_2 distance ρ . Define $H(t, y) = H_2(t, y) + (6L + 6)^{1/2}H_1(t)$. Then $\{H(t, y) : t \in E, y \in \mathbb{R}\}$ is sample bounded and uniformly continuous with respect to its L_2 distance d_H . Total boundedness of d_{H_1} and d_{H_2} implies that of d_H as can be seen from the equation

$$d_H((t_1, y_1), (t_2, y_2))^2 = d_{H_2}((t_1, y_1), (t_2, y_2))^2 + (6L + 6)d_{H_1}(t_1, t_2)^2.$$

Thus let T_0 be a countable dense subset in $(E \times \mathbb{R}, d_H)$. Since $d_{G_1} \leq d_H$ in view of (3.15) and (3.16), by the comparison theorem 2.3, $\{G_1(s, x) : (s, x) \in T_0\}$ is sample

bounded and uniformly continuous with respect to d_{G_1} . Since T_0 is also dense in $(E \times \mathbb{R}, d_{G_1})$, by Lemma 2.4, $\{G_1(s, x) : (s, x) \in E \times \mathbb{R}\}$ has a version which is sample bounded and uniformly d_{G_1} -continuous. \square

3.2. Local modulus. Recall that a positive function $L(x)$ defined on $(0, \infty)$ is slowly varying at infinity (in a neighborhood of zero) if $L(\lambda x)/L(x) \rightarrow 1, x \rightarrow \infty$ ($x \rightarrow 0$) for every $\lambda > 0$ (see [7, p. 276]). One says a function $U(x)$ is regularly varying at infinity (in a neighborhood of zero) if $U(x) = x^\rho L(x)$ for some $-\infty < \rho < \infty$, and some slowly varying at infinity (in a neighborhood of zero) function $L(x)$; ρ is called the exponent (see [7, p. 275]).

Lemma 3.13. *Let $w(x) > 0$ for $0 < x \leq 1/2$ and is regularly varying in a neighborhood of 0 with nonzero exponent α . Let $\theta_0 > 0$ be small enough such that $w(x)$ is non-increasing for $0 < x < \theta_0$. Then for $0 < \theta < \theta_0$*

$$\sum_{k=0}^{\infty} \frac{1}{w(2^{-k}\theta)^2} \leq \frac{C}{w(\theta)^2},$$

where C depends only on the weight function $w(x)$, but not on the argument x .

Proof. Since $w(x)$ is non-increasing for $0 < x < \theta_0$,

$$(\ln 2) \sum_{k=1}^{\infty} \frac{1}{w(2^{-k}\theta)^2} \leq \int_0^\theta \frac{1}{w(y)^2} \frac{dy}{y} \leq (\ln 2) \sum_{k=0}^{\infty} \frac{1}{w(2^{-k}\theta)^2}.$$

By Theorem 1 in [7, p. 281], we have

$$\frac{\frac{1}{w(\theta)^2}}{\int_0^\theta \frac{1}{w(y)^2} \frac{dy}{y}} \rightarrow \alpha, \quad \text{as } \theta \rightarrow 0,$$

where $\alpha > 0$ is the exponent of the regularly varying function $1/w(x)^2$ (note that if $w(x)$ is regularly varying, so is $1/w(x)^2$). Therefore, there is a constant $C(w)$ such that

$$\left| \frac{\int_0^\theta \frac{1}{w(y)^2} \frac{dy}{y}}{\frac{1}{w(\theta)^2}} \right| \leq C(w), \quad 0 < \theta < \theta_0. \quad \square$$

Lemma 3.14. *Given $\varepsilon > 0$, under the assumptions of Theorem 3.3, we have for $0 < a < b < 1$ and t fixed*

$$P^*(\exists s, \rho(s, t) \leq \varepsilon, \exists x \in (a, b] : X_s \leq x < X_t) \leq \frac{C\varepsilon^2}{w(b)^2} + (b - a),$$

and

$$P^*(\exists s, \rho(s, t) \leq \varepsilon, \exists x \in (a, b] : X_t \leq x < X_s) \leq \frac{C\varepsilon^2}{w(b)^2} + (b - a),$$

where C is a constant depending only on the function $w(x)$.

Proof. Let $N \geq 0$ be the biggest integer such that $b/2^N \geq a$. Then,

$$\begin{aligned}
& \mathbb{P}^*(\exists s, \rho(s, t) \leq \varepsilon, \exists x \in (a, b] : X_s \leq x < X_t) \\
& \leq \sum_{k=0}^{N-1} \mathbb{P}^*(\exists s, \rho(s, t) \leq \varepsilon, \exists x \in (2^{-k-1}b, 2^{-k}b] : X_s \leq x < X_t) \\
& \quad + \mathbb{P}^*(\exists s, \rho(s, t) \leq \varepsilon : X_s \leq x < X_t) \\
& \leq \sum_{k=0}^{N-1} \mathbb{P}^*(\exists s, \rho(s, t) \leq \varepsilon : X_s \leq 2^{-k}b < X_t) + \sum_{k=0}^{N-1} \mathbb{P}(2^{-k-1}b < X_t \leq 2^{-k}b) \\
& \quad + \mathbb{P}^*(\exists s, \rho(s, t) \leq \varepsilon : X_s \leq 2^{-N}b < X_t) + \mathbb{P}(a < X_t \leq 2^{-N}b) \\
& \leq \sum_{k=0}^{N-1} \mathbb{P}^*(\exists s, \rho(s, t) \leq \varepsilon : X_s \leq 2^{-k}b < X_t) + \sum_{k=0}^{N-1} (2^{-k}b - 2^{-k-1}b) \\
& \quad + \mathbb{P}^*(\exists s, \rho(s, t) \leq \varepsilon : X_s \leq 2^{-N}b < X_t) + 2^{-N}b - a \\
& \leq \sum_{k=0}^N \mathbb{P}^*(\exists s, \rho(s, t) \leq \varepsilon : X_s \leq 2^{-k}b < X_t) + \sum_{k=0}^{N-1} (2^{-k}b - 2^{-k-1}b) + 2^{-N}b - a \\
& \leq \sum_{k=0}^{\infty} \frac{L\varepsilon^2}{w(2^{-k}b)^2} + (b - a) \quad \text{using WL-condition to bound the probabilities} \\
& \leq \frac{C\varepsilon^2}{w(b)^2} + (b - a) \quad \text{by Lemma 3.13.}
\end{aligned}$$

The proof for the second part is similar; just change from $X_t \leq x < X_s$ for $2^{-k-1}b < x \leq 2^{-k}b$ to $X_t \leq 2^{-k-1}b < X_s$, with the same exceptional probability $(2^{-k}b - 2^{-k-1}b)$. \square

For the following, we use C to denote a constant which may change from line to line and depends only on the weight function $w(x)$.

Let the distance d be as in (3.10). Then,

$$e((s, x), (t, y)) := \max\{d(x, y), \rho(s, t)\}$$

is bounded by the Gaussian distance $(d(x, y)^2 + \rho(s, t)^2)^{1/2}$ on $E \times (0, 1)$ and will be used as the ' ρ ' in (iii) of Theorem 2.5.

Lemma 3.15. *For $t \in E$, $y \in (0, 1)$, let $x_0 := \inf\{x : \text{for some } s, e((s, x), (t, y)) < \varepsilon\}$, then*

$$(3.17) \quad d(x_0, y) \leq \varepsilon.$$

Proof. Indeed there exist a sequence $(s_n, x_n)_{n \in \mathbb{N}}$ in the set over which the infimum is taken such that $|x_n - x_0| \rightarrow 0$ as $n \rightarrow \infty$ and that $d(x_n, y) \leq \varepsilon$. By the sample continuity of the weighted Wiener process $w(x)W(x)$, we have $d(x_n, y) \rightarrow d(x_0, y)$ as $n \rightarrow \infty$. Hence we have obtained $d(x_0, y) \leq \varepsilon$. \square

Remark. The finiteness of $d(x_0, y)$ implies that x_0 can't be zero in view of (3.10) since $w(x) \rightarrow \infty$ and $xw(x)^2 \rightarrow 0$ as $x \rightarrow 0$.

Lemma 3.16. *For $t \in E$, $y \in (0, 1)$, let $x_1 := \sup\{x : \text{for some } s, e((s, x), (t, y)) < \varepsilon\}$, then*

$$(3.18) \quad d(y, x_1) \leq \varepsilon.$$

Proof. By a similar argument as in the proof of the previous lemma. \square

The following Lemma 3.17, Lemma 3.18 and Lemma 3.19 constitute a weighted version of Lemma 4 in [9]. For brevity of notation, for fixed $(t, y) \in E \times [0, 1]$, we write $(s, x) : e < \varepsilon, x \leq y$ for the set $\{(s, x) : e((s, x), (t, y)) < \varepsilon, x \leq y\}$.

Lemma 3.17. *Under the assumptions of Theorem 3.3, we have for all $\varepsilon > 0$ and $(t, y) \in E \times [0, 1]$,*

$$w(y)^2 \mathbf{P}^* \left(\sup_{(s,x): e < \varepsilon, x \leq y} |1_{X_s \leq x} - 1_{X_t \leq x}| > 0 \right) \leq C\varepsilon^2.$$

Proof. Let x_0 be as in Lemma 3.15. Then,

$$\begin{aligned} & w(y)^2 \mathbf{P}^* \left(\sup_{(s,x): e < \varepsilon, x \leq y} |1_{X_s \leq x} - 1_{X_t \leq x}| > 0 \right) \\ &= w(y)^2 \left(\mathbf{P}^*(\exists(s, x), e((s, x), (t, y)) < \varepsilon, x \leq y : X_s \leq x < X_t) \right. \\ &\quad \left. + \mathbf{P}^*(\exists(s, x), e((s, x), (t, y)) < \varepsilon, x \leq y : X_t \leq x < X_s) \right) \\ &= w(y)^2 \left(\mathbf{P}^*(\exists(s, x), e((s, x), (t, y)) < \varepsilon, x \in (x_0, y] : X_s \leq x < X_t) \right. \\ &\quad \left. + \mathbf{P}^*(\exists(s, x), e((s, x), (t, y)) < \varepsilon, x \in (x_0, y] : X_t \leq x < X_s) \right) \\ &\leq w(y)^2 (C\varepsilon^2/w(y)^2 + (y - x_0)) \text{ by Lemma 3.14} \\ &\leq C\varepsilon^2. \end{aligned}$$

For the last inequality, we used

$$w(y)^2(y - x_0) \leq d(x_0, y)^2 \leq \varepsilon^2 \quad \text{by (3.17).} \quad \square$$

Lemma 3.18. *Under the assumptions of Theorem 3.3, we have for all $\varepsilon > 0$ and $(t, y) \in E \times [0, 1]$,*

$$w(x_1)^2 \mathbf{P}^* \left(\sup_{(s,x): e < \varepsilon, x > y} |1_{X_s \leq x} - 1_{X_t \leq x}| > 0 \right) \leq C\varepsilon^2.$$

Proof. Let x_1 be as in Lemma 3.16. Then,

$$\begin{aligned} & w(x_1)^2 \mathbf{P}^* \left(\sup_{(s,x): e < \varepsilon, x > y} |1_{X_s \leq x} - 1_{X_t \leq x}| > 0 \right) \\ &= w(x_1)^2 \left(\mathbf{P}^*(\exists(s, x), e((s, x), (t, y)) < \varepsilon, x > y : X_s \leq x < X_t) \right. \\ &\quad \left. + \mathbf{P}^*(\exists(s, x), e((s, x), (t, y)) < \varepsilon, x > y : X_t \leq x < X_s) \right) \\ &= w(x_1)^2 \left(\mathbf{P}^*(\exists(s, x), e((s, x), (t, y)) < \varepsilon, x \in (y, x_1] : X_s \leq x < X_t) \right. \\ &\quad \left. + \mathbf{P}^*(\exists(s, x), e((s, x), (t, y)) < \varepsilon, x \in (y, x_1] : X_t \leq x < X_s) \right) \\ &\leq w(x_1)^2 (C\varepsilon^2/w(x_1)^2 + (x_1 - y)) \text{ by Lemma 3.14} \\ &\leq C\varepsilon^2. \end{aligned}$$

For the last inequality, we used

$$w(x_1)^2(x_1 - y) \leq d(y, x_1)^2 \leq \varepsilon^2 \quad \text{by (3.18).} \quad \square$$

In the following lemma, for fixed $(t, y) \in E \times [0, 1]$, we write $\sup_{(s,x):e<\varepsilon}$ for $\sup_{\{(s,x):e((s,x),(t,y))<\varepsilon\}}$ and the same applies to other similar quantities.

Lemma 3.19. *Under the assumptions of Theorem 3.3, we have for all $\varepsilon > 0$ and $(t, y) \in E \times [0, 1]$,*

$$\sup_{\alpha>0} \alpha^2 \mathbf{P}^* \left(\sup_{(s,x):e<\varepsilon} |w(x)1_{X_s \leq x} - w(y)1_{X_t \leq y}| > \alpha \right) \leq C\varepsilon^2.$$

Proof. We split the quantity:

$$w(x)1_{X_s \leq x} - w(y)1_{X_t \leq y} = [w(x)1_{X_t \leq x} - w(y)1_{X_t \leq y}] + [w(x)(1_{X_s \leq x} - 1_{X_t \leq x})].$$

Consider the weak L_2 norms of the components:

$$(3.19) \quad A := \sup_{\alpha>0} \alpha^2 \mathbf{P}^* \left(\sup_{(s,x):e<\varepsilon} |w(x)1_{X_t \leq x} - w(y)1_{X_t \leq y}| > \alpha \right)$$

$$(3.20) \quad B := \sup_{\alpha>0} \alpha^2 \mathbf{P}^* \left(\sup_{(s,x):e<\varepsilon} w(x)|1_{X_s \leq x} - 1_{X_t \leq x}| > \alpha \right).$$

First we estimate A. Since

$$\begin{aligned} \sup_{\alpha>0} \alpha^2 \mathbf{P}^* \left(\sup_{(s,x):e<\varepsilon} |w(x)1_{X_t \leq x} - w(y)1_{X_t \leq y}| > \alpha \right) \\ \leq \sup_{\alpha>0} \alpha^2 \mathbf{P}^* \left(\sup_{x:d(x,y)<\varepsilon} |w(x)1_{X_t \leq x} - w(y)1_{X_t \leq y}| > \alpha \right) \end{aligned}$$

and t is fixed, this is the case in Example 4.9 in [1]. Hence we have

$$(3.21) \quad A := \sup_{\alpha>0} \alpha^2 \mathbf{P}^* \left(\sup_{(s,x):e<\varepsilon} |w(x)1_{X_t \leq x} - w(y)1_{X_t \leq y}| > \alpha \right) \leq C\varepsilon^2.$$

Now we consider B. Since

$$\begin{aligned} \sup_{\alpha>0} \alpha^2 \mathbf{P}^* \left(\sup_{(s,x):e<\varepsilon} w(x)|1_{X_s \leq x} - 1_{X_t \leq x}| > \alpha \right) &\leq \sup_{\alpha>0} \alpha^2 \mathbf{P}^* \left(\sup_{(s,x):e<\varepsilon, x \leq y} w(x)|1_{X_s \leq x} - 1_{X_t \leq x}| > \alpha \right) \\ &\quad + \sup_{\alpha>0} \alpha^2 \mathbf{P}^* \left(\sup_{(s,x):e<\varepsilon, x > y} w(x)|1_{X_s \leq x} - 1_{X_t \leq x}| > \alpha \right), \end{aligned}$$

it suffices to consider bounds of the last two quantities. Without loss of generality, we assume $w(x)$ is monotone on $(0, 1/2]$. For $\alpha > 0$, let

$$x_\alpha = \sup\{x \in [0, 1/2] : w(x) > \alpha\}.$$

Case $x \leq y$.

Recall $x_0 = \inf\{x : e((s, x), (t, y)) < \varepsilon\}$. First we consider the extreme cases for x_α .

(1). By continuity of $w(\cdot)$, if $x_\alpha > y$, then $\alpha \leq w(y)$, consequently

$$\begin{aligned} \sup_{\alpha < w(y)} \alpha^2 \mathbf{P}^* \left(\sup_{(s,x):e<\varepsilon, x \leq y} w(x)|1_{X_s \leq x} - 1_{X_t \leq x}| > \alpha \right) \\ \leq w(y)^2 \mathbf{P}^* \left(\sup_{(s,x):e<\varepsilon, x \leq y} |1_{X_s \leq x} - 1_{X_t \leq x}| > 0 \right) \leq C\varepsilon^2 \text{ by Lemma 3.17.} \end{aligned}$$

(2). If $x_\alpha \leq x_0$, then $w(x_0) \leq \alpha$, hence $w(x) \leq \alpha$ for $x_0 \leq x$. For α such that $x_\alpha \leq x_0$, the event under the probability of (3.20) is empty.

(3). Now $x_0 < x_\alpha \leq y$. In this case, $w(y) \leq \alpha < w(x_0)$. Take $\varepsilon > 0$. We have

$$\begin{aligned}
B &:= \sup_{w(y) \leq \alpha < w(x_0)} \alpha^2 \mathbf{P}^* \left(\sup_{(s,x): e < \varepsilon} w(x) |1_{X_s \leq x} - 1_{X_t \leq x}| > \alpha \right) \\
&\leq \sup_{w(y) \leq \alpha < w(x_0)} \alpha^2 \mathbf{P}^* \left(\sup_{(s,x): e < \varepsilon} w(x) 1_{X_s \leq x < X_t} > \alpha \right) \\
&\quad + \sup_{w(y) \leq \alpha < w(x_0)} \alpha^2 \mathbf{P}^* \left(\sup_{(s,x): e < \varepsilon} w(x) 1_{X_t \leq x < X_s} > \alpha \right) \\
&= I + II.
\end{aligned}$$

For I ,

$$\begin{aligned}
I &= \sup_{w(y) \leq \alpha < w(x_0)} \alpha^2 \mathbf{P}^* \left(\sup_{(s,x): e < \varepsilon} w(x) 1_{X_s \leq x < X_t} > \alpha \right) \\
&\leq \sup_{x_0 < x_\alpha \leq y} w(x_\alpha)^2 \mathbf{P}^* \left(\sup_{(s,x): e < \varepsilon} w(x) 1_{X_s \leq x < X_t} > \alpha \right) \\
&\leq \sup_{x_0 < x_\alpha \leq y} w(x_\alpha)^2 \mathbf{P}^* \left(\sup_{(s,x): e < \varepsilon} 1_{X_s \leq x < X_t, x \leq x_\alpha} > 0 \right) \\
&\leq \sup_{x_0 < x_\alpha \leq y} w(x_\alpha)^2 (C\varepsilon^2/w(x_\alpha)^2 + (x_\alpha - x_0)) \text{ using Lemma 3.14} \\
&\leq C\varepsilon^2.
\end{aligned}$$

For the last inequality, we used

$$w(x_\alpha)^2(x_\alpha - x_0) \leq d(x_0, x_\alpha)^2 \leq d(x_0, y)^2 \leq \varepsilon^2$$

of Lemma 3.9 and Lemma 3.15.

II can be handled in the same way.

Case $x > y$.

Recall $x_1 = \sup\{x : e((s, x), (t, y)) < \varepsilon\}$. First we consider the extreme cases for x_α .

(1). By continuity of $w(\cdot)$, if $x_\alpha > x_1$, then $\alpha \leq w(x_1)$, consequently

$$\begin{aligned}
&\sup_{\alpha < w(x_1)} \alpha^2 \mathbf{P}^* \left(\sup_{(s,x): e < \varepsilon, x > y} w(x) |1_{X_s \leq x} - 1_{X_t \leq x}| > \alpha \right) \\
&\leq w(x_1)^2 \mathbf{P}^* \left(\sup_{(s,x): e < \varepsilon, x > y} |1_{X_s \leq x} - 1_{X_t \leq x}| > 0 \right) \leq C\varepsilon^2.
\end{aligned}$$

by Lemma 3.18. consider $\alpha \geq w(y)$, i.e. $x_\alpha \leq y$.

(2). If $x_\alpha \leq y$, then $w(y) \leq \alpha$, hence $w(x) \leq \alpha$ for $y \leq x$. For α such that $x_\alpha \leq y$, the event under the probability of (3.20) is empty.

(3). Now $y < x_\alpha \leq x_1$. In this case, $w(x_1) \leq \alpha < w(y)$. Take $\varepsilon > 0$. We have

$$\begin{aligned}
B &:= \sup_{w(x_1) \leq \alpha < w(y)} \alpha^2 \mathbf{P}^* \left(\sup_{(s,x): e < \varepsilon} w(x) |1_{X_s \leq x} - 1_{X_t \leq x}| > \alpha \right) \\
&\leq \sup_{w(x_1) \leq \alpha < w(y)} \alpha^2 \mathbf{P}^* \left(\sup_{(s,x): e < \varepsilon} w(x) 1_{X_s \leq x < X_t} > \alpha \right) \\
&\quad + \sup_{w(x_1) \leq \alpha < w(y)} \alpha^2 \mathbf{P}^* \left(\sup_{(s,x): e < \varepsilon} w(x) 1_{X_t \leq x < X_s} > \alpha \right) \\
&= I + II.
\end{aligned}$$

For I ,

$$\begin{aligned}
I &= \sup_{w(x_1) \leq \alpha < w(y)} \alpha^2 \mathbf{P}^* \left(\sup_{(s,x): e < \varepsilon} w(x) 1_{X_s \leq x < X_t} > \alpha \right) \\
&\leq \sup_{y < x_\alpha \leq x_1} w(x_\alpha)^2 \mathbf{P}^* \left(\sup_{(s,x): e < \varepsilon} w(x) 1_{X_s \leq x < X_t} > \alpha \right) \\
&\leq \sup_{y < x_\alpha \leq x_1} w(x_\alpha)^2 \mathbf{P}^* \left(\sup_{(s,x): e < \varepsilon} 1_{X_s \leq x < X_t, x \leq x_\alpha} > 0 \right) \\
&\leq \sup_{y < x_\alpha \leq x_1} w(x_\alpha)^2 (C\varepsilon^2 / w(x_\alpha)^2 + (x_\alpha - y)) \text{ using Lemma 3.14} \\
&\leq C\varepsilon^2.
\end{aligned}$$

For the last inequality, we used

$$w(x_\alpha)^2(x_\alpha - y) \leq d(y, x_\alpha)^2 \leq d(y, x_1)^2 \leq \varepsilon^2$$

of Lemma 3.9 and Lemma 3.16.

II can be handled in the same way. Hence we have $B \leq C\varepsilon^2$. This together with (3.21) completes the proof. \square

Proof of Theorem 3.3. We apply Theorem 2.5 to the process $\{w(y)(1_{X_t \leq y} - y) : t \in E, y \in [0, 1]\}$.

Since for each $s \in E$, $X(s)$ takes values on $(0, 1)$ and $xw(x) \rightarrow 0$ as $x \rightarrow 0$, almost surely

$$\sup_{s \in E, x \in [0, 1/2]} w(x) |1_{X_s \leq x} - x| < \infty.$$

Also we observe for each $s \in E, x \in [0, 1/2]$

$$\mathbf{P}(w(x)(1_{X_s \leq x} - x))^2 < \infty.$$

Since $w(x)$ is decreasing near 0,

$$\begin{aligned}
\lim_{\alpha \rightarrow \infty} \alpha^2 \mathbf{P}^* \left(\sup_{s \in E, x \in (0, 1/2]} w(x) 1_{X_s \leq x} > \alpha \right) &\leq \lim_{\alpha \rightarrow \infty} \alpha^2 \mathbf{P}^* \left(\sup_{s \in E} w(X_s) > \alpha \right) \\
&= 0 \text{ by assumption of Theorem 3.3,}
\end{aligned}$$

which in turn implies

$$\lim_{\alpha \rightarrow \infty} \alpha^2 \mathbf{P}^* \left(\sup_{s \in E, x \in (0, 1/2]} w(x) |1_{X_s \leq x} - x| > \alpha \right) = 0.$$

This verifies (i) in Theorem 2.5. Corollary 3.11 verifies the pre-Gaussian condition (ii).

In view of Lemma 3.19 and the inequality

$$\Lambda_{2,\infty}(f + g) \leq C(\Lambda_{2,\infty}(f) + \Lambda_{2,\infty}(g))$$

where $\Lambda_{2,\infty}(f) := [\sup_{t>0} t^2 \mathbf{P}(\{|f| > t\})]^{1/2}$ for some constant C , to verify the local modulus condition (iii) in Theorem 2.5 for the functions $w(x)(1_{X_s \leq x} - x)$, it is enough to have

$$(3.22) \quad \sup_{\alpha > 0} \alpha^2 \mathbf{P}^* \left(\sup_{d(x,y) \leq \epsilon} |w(x)x - w(y)y| > \alpha \right) \leq K\epsilon^2$$

for some constant K . W.o.l.g, assume $x < y$. Inequality 3.22 follows from

$$\begin{aligned}
|xw(x) - yw(y)|^2 &\leq 2x^2(w(x) - w(y))^2 + 2w(y)^2(y - x)^2 \\
&\leq 2x(w(x) - w(y))^2 + 2w(y)^2(y - x) \\
&= 2d(x, y)^2 \text{ by 3.10} \\
&\leq 2\epsilon^2.
\end{aligned}$$

□

4. AN EXAMPLE

A special class of uniform processes (copula processes) can be obtained from distributional transforms. Specifically, given a process $Y := \{Y_t : t \in E\}$, define $X := X_t := \tilde{F}_t(Y_t)$, where $\tilde{F}_t(\cdot)$ is the distributional transform of the df of Y_t . Now, we give an example as an application of Theorem 3.3 when $\{Y_t : t \in E\} = \{B_t : t \in [1, 2]\}$, where B_t is a Brownian motion.

Theorem 4.1. *Let $\{B_t : t \geq 0\}$ be a Brownian motion and $F_t(x)$ be the distribution function of B_t . Let $w(x) = x^{-\alpha}L(x)$, for $0 < x < 1/2$, $0 < \alpha < 1/2$, and $L(x)$ slowly varying at 0 and assume $w(x)$ is symmetric about $1/2$. Further assume that $w(x)$ is non-increasing and $xw(x)^2$ non-decreasing near 0. Then*

$$\{w(F_t(y))(1_{B_t \leq y} - F_t(y)) : t \in [1, 2], y \in \mathbb{R}\} \in \text{CLT in } \ell^\infty([1, 2] \times \mathbb{R}).$$

Remarks 4.2. The interval $[1, 2]$ can be replaced by any interval $[a, b]$ provided $a > 0$, which can be seen from the proof of the above theorem; also a priori, we need $F_t(\cdot)$ be strictly increasing.

We will verify the conditions in Proposition 3.5 to prove this theorem at the end of this section. To this end, we start with some lemmas. For the following, let $\phi(x) = (2\pi)^{-1/2}e^{-x^2/2}$ and $\Phi(y) := (2\pi)^{-1/2} \int_{-\infty}^y e^{-s^2/2} ds$.

Lemma 4.3 ([6], p. 175). *For $y > 0$,*

$$y^{-1}(1 - y^{-2})(2\pi)^{-1/2}e^{-y^2/2} \leq \Phi(-y) \leq y^{-1}(2\pi)^{-1/2}e^{-y^2/2}.$$

In particular, for $y > \sqrt{2}$,

$$2^{-1}y^{-1}(2\pi)^{-1/2}e^{-y^2/2} \leq \Phi(-y) \leq y^{-1}(2\pi)^{-1/2}e^{-y^2/2}.$$

Lemma 4.4 ([14], p. 18). *Let $L(x)$ be a slowly varying function at 0, then for any $\gamma > 0$,*

$$x^\gamma L(x) \rightarrow 0, x^{-\gamma} L(x) \rightarrow \infty \text{ as } x \rightarrow 0.$$

Consequently, for $0 < \gamma_1 < 2\alpha < \gamma_2 < 1$ and a function $L(x)$ slowly varying (at 0), there are constants c_1, c_2 ,

$$c_1 x^{\gamma_2} \leq x^{2\alpha}/L(x) \leq c_2 x^{\gamma_1}, \quad 0 < x < 1/2.$$

For $c > 0$, let $L_c(x) = \exp(c\sqrt{\ln(1/x)})$.

Lemma 4.5. *The function $L_c(x)$ is slowly varying at 0; that is for all $\lambda > 0$*

$$\lim_{x \rightarrow 0} \frac{L_c(\lambda x)}{L_c(x)} = 1.$$

Proof. By definition. □

Lemma 4.6. For $0 < x < 1/4$, let $y = -\Phi^{-1}(x)$. Then

$$y \leq \sqrt{2 \ln(1/x)}$$

and

$$\begin{aligned} \phi(-\Phi^{-1}(x) + c) &\leq CxL_C(x) && \text{for } c < 0, \\ \phi(-\Phi^{-1}(x) + c) &\leq 2^{3/2}x\sqrt{\ln(1/x)} && \text{for } c \geq 0, \end{aligned}$$

where C depends only on c .

Proof. By Lemma 4.3, for $y > (2\pi)^{-1/2}$, $x \leq e^{-y^2/2}$; hence $y \leq \sqrt{2 \ln(1/x)}$.

$$\begin{aligned} \phi(-\Phi^{-1}(x) + c) &= (2\pi)^{-1/2} \exp(-\frac{(y+c)^2}{2}) \\ &= (2\pi)^{-1/2} \exp(-\frac{y^2}{2}) \exp(-yc) \exp(-c^2/2) \\ &\leq 2y\Phi(-y) \exp(-yc) && \text{by Lemma 4.3} \\ &\leq 2xy \exp(-yc). \end{aligned}$$

The statement for $c > 0$ follows from that $\exp(-yc) \leq 1$ and $y \leq \sqrt{2 \ln(1/x)}$. For $c \leq 0$ the statement follows from that $y \leq C \exp(yC)$ for some constant C . \square

Theorem 4.7 (Borell, see also [10], Theorem 7.1). Let $G = (G_t)_{t \in T}$ be a centered Gaussian process indexed by a countable set T such that $\sup_{t \in T} G_t < \infty$ almost surely. Then, $E(\sup_{t \in T} G_t) < \infty$ and for every $r > 0$

$$P(\{\sup_{t \in T} G_t \geq E(\sup_{t \in T} G_t) + r\}) \leq e^{-r^2/2\sigma^2},$$

where $\sigma = \sup_{t \in T} (EG_t^2)^{1/2}$.

For the following, let B_t be a Brownian motion and $F_t(x)$ the distribution function of B_t , which is $\Phi(\frac{x}{\sqrt{t}})$. Also for $1 \leq t \leq 2$, $0 < \varepsilon < 1/2$, set

$$\begin{aligned} D &:= D(t, \varepsilon) := \sup_{t < s \leq t+\varepsilon} \frac{B_s - B_t}{\sqrt{s}}, \\ m &:= m(t, \varepsilon) := E \sup_{t < s \leq t+\varepsilon} \frac{B_s - B_t}{\sqrt{s}}, \\ m_0 &:= \sup\{m(t, \varepsilon) : 1 \leq t \leq 2, 0 < \varepsilon < 1/2\}. \end{aligned}$$

We use C to denote a constant, which may vary in each occurrence.

Lemma 4.8. For $1 \leq t \leq 2$, $0 < \varepsilon < 1/2$

$$m \leq 2(2/\pi)^{1/2} \varepsilon^{1/2}.$$

Proof. By the maximal inequality for Brownian motion,

$$\begin{aligned} m &:= E \sup_{t < s \leq t+\varepsilon} \frac{B_s - B_t}{\sqrt{s}} \\ &\leq E \sup_{t < s \leq t+\varepsilon} \frac{|B_s - B_t|}{\sqrt{t}} \\ &\leq E \varepsilon^{1/2} 2|N(0, 1)| \\ &\leq 2(2/\pi)^{1/2} \varepsilon^{1/2}. \end{aligned} \quad \square$$

Lemma 4.9. Let $d := E(\sup_{1 \leq t \leq 2} \frac{B_t}{\sqrt{t}})$. Then, $d > 0$ and

$$P(\inf_{1 \leq t \leq 2} F_t(B_t) \leq x) \leq (2\pi)^{1/2} \phi(-\Phi^{-1}(x) - d).$$

Proof.

$$\begin{aligned}
P(\inf_{1 \leq t \leq 2} F_t(B_t) \leq x) &= P(\inf_{1 \leq t \leq 2} \frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x)) \\
&= P(\sup_{1 \leq t \leq 2} \frac{-B_t}{\sqrt{t}} \geq -\Phi^{-1}(x)) \\
&= P(\sup_{1 \leq t \leq 2} \frac{-B_t}{\sqrt{t}} \geq d - \Phi^{-1}(x) - d)
\end{aligned}$$

which, by Theorem 4.7 and for x such that $-\Phi^{-1}(x) - d > 0$, is

$$\begin{aligned}
&\leq \exp(-\frac{(-\Phi^{-1}(x)-d)^2}{2}) \\
&= (2\pi)^{1/2} \phi(-\Phi^{-1}(x) - d).
\end{aligned}$$

Note that here $\sigma^2 = \sup_{1 \leq t \leq 2} E(\frac{-B_t}{\sqrt{t}})^2 = 1$. \square

Lemma 4.10. *Let $w(x) = x^{-\alpha}L(x)$, $0 < \alpha < 1/2$ and $L(x)$ be a slowly varying function (growing to infinity as $x \downarrow 0$). Assume $w(x)$ is decreasing near 0. Then*

$$\lim_{\lambda \rightarrow \infty} \lambda^2 P^*(\sup_{1 \leq t \leq 2} w(F_t(B_t)) > \lambda) = 0.$$

Proof. Let $\lambda = w(x)$. Then, by Lemma 4.6 and Lemma 4.9,

$$\begin{aligned}
\lim_{\lambda \rightarrow \infty} \lambda^2 P^*(\sup_{1 \leq t \leq 2} w(F_t(B_t)) > \lambda) &= \lim_{\lambda \rightarrow \infty} \lambda^2 P^*(w(\inf_{1 \leq t \leq 2} F_t(B_t)) > \lambda) \\
&= \lim_{x \rightarrow 0} w(x)^2 P^*(\inf_{1 \leq t \leq 2} F_t(B_t) \leq x) \\
&\leq \lim_{x \rightarrow 0} w(x)^2 (2\pi)^{1/2} \phi(-\Phi^{-1}(x) - d) \\
&\leq \lim_{x \rightarrow 0} x^{-2\alpha} L(x)^2 (2\pi)^{1/2} C x L_C(x) \\
&= 0.
\end{aligned}$$

\square

Lemma 4.11. *For $1 \leq t \leq 2$, $0 < \varepsilon < 1/2$, and $l > m$,*

$$P(\frac{B_t}{\sqrt{t}} < l \leq \sup_{t < s \leq t+\varepsilon} \frac{B_s}{\sqrt{s}}) \leq C_t \varepsilon^{1/2} \phi(l - m)^{\frac{t+\varepsilon}{t+2\varepsilon}},$$

where C_t is a constant depending only on t . In particular, if we let $C := \sup_{1 \leq t \leq 2} C_t$, and recall $m_0 := \sup\{m(t, \varepsilon) : 1 \leq t \leq 2, 0 < \varepsilon < 1/2\}$, then for $l > m_0$, we have $C < \infty$ and

$$(4.1) \quad P(\frac{B_t}{\sqrt{t}} < l \leq \sup_{t < s \leq t+\varepsilon} \frac{B_s}{\sqrt{s}}) \leq C \varepsilon^{1/2} \phi(l - m_0)^{\frac{t+\varepsilon}{t+2\varepsilon}}.$$

Proof. Since $\sigma^2 := \sup_{t < s \leq t+\varepsilon} E(\frac{B_s - B_t}{\sqrt{s}})^2 = \frac{\varepsilon}{t+\varepsilon}$, by Borell's concentration inequality Theorem [4.7] (since the process $(B_s - B_t)/s^{1/2}$ is continuous in s , we can take supremum over a countable set in the definition of D) it follows that for $r > 0$

$$(4.2) \quad P(D > m + r) \leq e^{-r^2(t+\varepsilon)/(2\varepsilon)}.$$

Hence, conditioning on $\frac{B_t}{\sqrt{t}}$,

$$\begin{aligned}
P(\frac{B_t}{\sqrt{t}} < l \leq \sup_{t < s \leq t+\varepsilon} \frac{B_s}{\sqrt{s}}) &\leq P(\frac{B_t}{\sqrt{t}} < l \leq \sup_{t < s \leq t+\varepsilon} (\frac{B_s}{\sqrt{s}} - \frac{B_t}{\sqrt{t}}) + \sup_{t < s \leq t+\varepsilon} \frac{B_t}{\sqrt{s}}) \\
&= E_{\frac{B_t}{\sqrt{t}}} P(\frac{B_t}{\sqrt{t}} < l \leq \sup_{t < s \leq t+\varepsilon} (\frac{B_s}{\sqrt{s}} - \frac{B_t}{\sqrt{t}}) + \sup_{t < s \leq t+\varepsilon} \frac{B_t}{\sqrt{s}} | \frac{B_t}{\sqrt{t}})
\end{aligned}$$

by independence of $\{B_s - B_t : s > t\}$ and B_t

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \mathbb{P}(y < l \leq D + \sup_{t < s \leq t+\varepsilon} \{(t/s)^{1/2}y\}) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\
&= \int_0^{\infty} \mathbb{P}(y < l \leq D + \sup_{t < s \leq t+\varepsilon} \{(t/s)^{1/2}y\}) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\
&\quad + \int_{-\infty}^0 \mathbb{P}(y < l \leq D + \sup_{t < s \leq t+\varepsilon} \{(t/s)^{1/2}y\}) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\
(4.3) \quad &= I + II.
\end{aligned}$$

Note that

$$\begin{aligned}
&\sup_{t < s \leq t+\varepsilon} \{(t/s)^{1/2}y\} = y \quad \text{for } y > 0, \\
&\sup_{t < s \leq t+\varepsilon} \{(t/s)^{1/2}y\} = ((t/(t+\varepsilon))^{1/2}y =: ay \quad \text{for } y \leq 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
I &= \int_0^{\infty} \mathbb{P}(y < l \leq D + y) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\
&= \int_0^{l-m} \mathbb{P}(y < l \leq D + y) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy + \int_{l-m}^l \mathbb{P}(y < l \leq D + y) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\
&= \int_0^{l-m} \mathbb{P}(D \geq l - y) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy + \int_{l-m}^l \mathbb{P}(y < l \leq D + y) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy
\end{aligned}$$

by inequality (4.2) for the first summand and noting $r := l - y - m > 0$

$$\leq \int_0^{l-m} e^{-\frac{(l-y-m)^2(t+\varepsilon)}{2\varepsilon}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy + \int_{l-m}^l \mathbb{P}(y < l \leq D + y) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

by completing the square in y for the first summand

$$\leq \left(\frac{\varepsilon}{t+2\varepsilon}\right)^{1/2} e^{-\frac{(l-m)^2}{2} \frac{t+\varepsilon}{t+2\varepsilon}} + m\phi(l-m)$$

bounding m using Lemma 4.8

$$\begin{aligned}
(4.4) \quad &\leq \left(\frac{\varepsilon}{t+2\varepsilon}\right)^{1/2} (2\pi)^{1/2} \phi(l-m) \frac{t+\varepsilon}{t+2\varepsilon} + 2(2/\pi)^{1/2} \varepsilon^{1/2} \phi(l-m).
\end{aligned}$$

For II ,

$$\begin{aligned}
II &= \int_{-\infty}^0 \mathbb{P}(y < l \leq D + ay) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\
&\leq \int_{-\infty}^0 \mathbb{P}(D \geq l - ay) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy
\end{aligned}$$

by equation (4.2)

$$\leq \int_{-\infty}^0 e^{-\frac{(l-ay-m)^2(t+\varepsilon)}{2\varepsilon}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

by completing the square in y

$$(4.5) \quad \begin{aligned} &\leq \left(\frac{\varepsilon}{t+\varepsilon}\right)^{1/2} e^{-\frac{(l-m)^2}{2}} \\ &= \left(\frac{\varepsilon}{t+\varepsilon}\right)^{1/2} (2\pi)^{1/2} \phi(l-m). \end{aligned}$$

Combining (4.3), (4.4), and (4.5) completes the proof. \square

Lemma 4.12. *For $1 \leq t \leq 2$, $0 < \varepsilon \leq 1/2$, there is a universal constant C such that for $0 < x < 1/4$*

$$\mathbb{P}\left(\frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) < \sup_{t < s \leq t+\varepsilon} \frac{B_s}{\sqrt{s}}\right) \leq C\varepsilon^{1/2} \left(x \ln \frac{1}{x}\right).$$

Proof.

$$\begin{aligned} &\mathbb{P}\left(\frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) < \sup_{t < s \leq t+\varepsilon} \frac{B_s}{\sqrt{s}}\right) \\ &= \mathbb{P}\left(\frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) < \left[\sup_{t < s \leq t+\varepsilon} \frac{B_s}{\sqrt{s}} - \frac{B_t}{\sqrt{s}}\right] + \sup_{t < s \leq t+\varepsilon} \frac{B_t}{\sqrt{s}}\right) \end{aligned}$$

letting $D = \sup_{t < s \leq t+\varepsilon} \frac{B_s}{\sqrt{s}} - \frac{B_t}{\sqrt{s}}$ and noting $B_t < 0$ inside the probability above

$$\leq \mathbb{E}_D \mathbb{P}\left(\frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) \leq D + \frac{B_t}{\sqrt{t+\varepsilon}} \mid D\right)$$

by independence of $\{B_s - B_t : s > t\}$ and B_t

$$\begin{aligned} &= \mathbb{E}_D \mathbb{P}\left(\frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) \leq \frac{B_t}{\sqrt{t+\varepsilon}} + D\right) \\ &= \mathbb{E}_D \mathbb{P}\left(\left(\frac{t+\varepsilon}{t}\right)^{1/2} (\Phi^{-1}(x) - D) \leq \frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x)\right) \end{aligned}$$

bounding the density of $\frac{B_t}{\sqrt{t}}$ from above by $\phi(\Phi^{-1}(x))$

$$\begin{aligned} &\leq \mathbb{E}_D \phi(\Phi^{-1}(x)) \left[\left(1 - \left(\frac{t+\varepsilon}{t}\right)^{1/2}\right) \Phi^{-1}(x) + \left(\frac{t+\varepsilon}{t}\right)^{1/2} D \right] \\ &\leq \phi(\Phi^{-1}(x)) (-\Phi^{-1}(x)) (\varepsilon/t) + \phi(-\Phi^{-1}(x)) \left(\frac{t+\varepsilon}{t}\right)^{1/2} \mathbb{E}_D D \\ &\leq C \left(x \ln \frac{1}{x}\right) (\varepsilon/t) + C x \left(\ln \frac{1}{x}\right)^{1/2} 8\varepsilon^{1/2} \text{ by Lemma 4.6 and Lemma 4.8} \\ &\leq C\varepsilon^{1/2} \left(x \ln \frac{1}{x}\right). \end{aligned} \quad \square$$

Proposition 4.13. *For $1 \leq t \leq 2$, $0 < \varepsilon \leq 1/2$, there is a universal constant C such that for $0 < x < 1/4$*

$$\mathbb{P}(F_t(B_t) \leq x < \sup_{s: |s-t| \leq \varepsilon} F_s(B_s)) \leq C\varepsilon^{1/2} \left(x \ln \frac{1}{x}\right) + C\varepsilon^{1/2} \phi(-\Phi^{-1}(x) - m_0) \frac{t}{t+\varepsilon}.$$

Proof.

$$\begin{aligned} &\mathbb{P}(F_t(B_t) \leq x < \sup_{\{s: |s-t| \leq \varepsilon\}} F_s(B_s)) \\ &= \mathbb{P}\left(\Phi\left(\frac{B_t}{\sqrt{t}}\right) \leq x < \sup_{\{s: |s-t| \leq \varepsilon\}} \Phi\left(\frac{B_s}{\sqrt{s}}\right)\right) \\ &= \mathbb{P}\left(\frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) < \sup_{\{s: |s-t| \leq \varepsilon\}} \frac{B_s}{\sqrt{s}}\right) \\ &\leq \mathbb{P}\left(\frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) < \sup_{t < s \leq t+\varepsilon} \frac{B_s}{\sqrt{s}}\right) + \mathbb{P}\left(\frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) < \sup_{t-\varepsilon \leq s < t} \frac{B_s}{\sqrt{s}}\right) \\ &= I + II. \end{aligned}$$

By Lemma 4.12,

$$(4.6) \quad I \leq C\varepsilon^{1/2}(x \ln \frac{1}{x}).$$

Now we consider II .

$$\begin{aligned}
II &= P(\frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) < \sup_{t-\varepsilon \leq s < t} \frac{B_s}{\sqrt{s}}) \\
&= P(\frac{B_{t-\varepsilon}}{\sqrt{t-\varepsilon}} \leq \Phi^{-1}(x), \frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) < \sup_{t-\varepsilon \leq s < t} \frac{B_s}{\sqrt{s}}) \\
&\quad + P(\frac{B_{t-\varepsilon}}{\sqrt{t-\varepsilon}} > \Phi^{-1}(x), \frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) < \sup_{t-\varepsilon \leq s < t} \frac{B_s}{\sqrt{s}}) \\
&\leq P(\frac{B_{t-\varepsilon}}{\sqrt{t-\varepsilon}} \leq \Phi^{-1}(x) < \sup_{t-\varepsilon < s \leq t} \frac{B_s}{\sqrt{s}}) + P(\frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) < \frac{B_{t-\varepsilon}}{\sqrt{t-\varepsilon}}) \\
&\leq P(\frac{B_{t-\varepsilon}}{\sqrt{t-\varepsilon}} \leq \Phi^{-1}(x) < \sup_{t-\varepsilon < s \leq t} \frac{B_s}{\sqrt{s}}) + P(\frac{B_{t-\varepsilon}}{\sqrt{t-\varepsilon}} \leq -\Phi^{-1}(x) < \frac{B_t}{\sqrt{t}}) \\
(4.7) \quad &\leq C\varepsilon^{1/2}(x \ln \frac{1}{x}) + C\varepsilon^{1/2}\phi(-\Phi^{-1}(x) - m_0)^{\frac{t}{t+\varepsilon}} \quad \text{by Lemmas 4.12 and 4.11.}
\end{aligned}$$

□

Proposition 4.14. *For $1 \leq t \leq 2$, $0 < \varepsilon \leq 1/2$, there is a universal constant C such that for $0 < x < 1/4$*

$$P(\inf_{\{s: |s-t| \leq \varepsilon\}} F_s(B_s) \leq x < F_t(B_t)) \leq C\varepsilon^{1/2}\phi(-\Phi^{-1}(x) - m_0)^{\frac{t}{t+\varepsilon}} + C\varepsilon^{1/2}(x \ln \frac{1}{x}).$$

Proof. First we consider the case $\{s > t : |s-t| \leq \varepsilon\}$. Let $D = \sup_{t < s \leq t+\varepsilon} \frac{B_s}{\sqrt{s}} - \frac{B_t}{\sqrt{t}}$.

$$\begin{aligned}
&P(\inf_{t < s \leq t+\varepsilon} F_s(B_s) \leq x < F_t(B_t)) \\
&= P(\inf_{t < s \leq t+\varepsilon} \frac{B_s}{\sqrt{s}} \leq \Phi^{-1}(x) < \frac{B_t}{\sqrt{t}}) \\
&= P(\frac{B_t}{\sqrt{t}} < -\Phi^{-1}(x) \leq \sup_{t < s \leq t+\varepsilon} \frac{B_s}{\sqrt{s}}) \\
&\leq C\varepsilon^{1/2}\phi(-\Phi(x) - m_0)^{\frac{t+\varepsilon}{t+2\varepsilon}} \quad \text{by Lemma 4.11.}
\end{aligned}$$

For the the case $\{s < t : |s-t| \leq \varepsilon\}$,

$$\begin{aligned}
&P(\inf_{t-\varepsilon \leq s < t} F_s(B_s) \leq x < F_t(B_t)) \\
&= P(\frac{B_t}{\sqrt{t}} < -\Phi^{-1}(x) \leq \sup_{t-\varepsilon \leq s < t} \frac{B_s}{\sqrt{s}}) \\
&= P(\frac{B_{t-\varepsilon}}{\sqrt{t-\varepsilon}} < -\Phi^{-1}(x), \frac{B_t}{\sqrt{t}} < -\Phi^{-1}(x) \leq \sup_{t-\varepsilon \leq s < t} \frac{B_s}{\sqrt{s}}) \\
&\quad + P(\frac{B_{t-\varepsilon}}{\sqrt{t-\varepsilon}} \geq -\Phi^{-1}(x), \frac{B_t}{\sqrt{t}} < -\Phi^{-1}(x) \leq \sup_{t-\varepsilon \leq s < t} \frac{B_s}{\sqrt{s}}) \\
&= P(\frac{B_{t-\varepsilon}}{\sqrt{t-\varepsilon}} < -\Phi^{-1}(x) \leq \sup_{t-\varepsilon \leq s < t} \frac{B_s}{\sqrt{s}}) + P(\frac{B_t}{\sqrt{t}} < -\Phi^{-1}(x) \leq \frac{B_{t-\varepsilon}}{\sqrt{t-\varepsilon}}) \\
&= P(\frac{B_{t-\varepsilon}}{\sqrt{t-\varepsilon}} < -\Phi^{-1}(x) \leq \sup_{t-\varepsilon \leq s < t} \frac{B_s}{\sqrt{s}}) + P(\frac{B_{t-\varepsilon}}{\sqrt{t-\varepsilon}} \leq \Phi^{-1}(x) < \frac{B_t}{\sqrt{t}}) \\
&\leq C\varepsilon^{1/2}\phi(-\Phi(x) - m_0)^{\frac{t}{t+\varepsilon}} + C\varepsilon^{1/2}(x \ln \frac{1}{x}) \quad \text{by Lemmas 4.11 and 4.12.} \quad \square
\end{aligned}$$

Proof of Theorem 4.1. Let $0 < \varepsilon < 1/2$ and $1 \leq t \leq 2$. Choose $\theta > 4$ big enough such that $\frac{t}{t+\varepsilon^\theta} > 2\alpha$ uniformly in t and ε . Let $\rho(s, t) = |s - t|^{1/\theta}$. Then $\rho(s, t)$ is a continuous Gaussian metric on $[0, 1]$ (indeed it is the L_2 distance of the fractional Brownian motion with Hurst index $1/\theta$). By Lemmas 4.4, 4.5, and 4.6, it follows that for $0 < x < 1/4$ (for $1/4 \leq x \leq 1/2$, the proof is trivial as $w(\cdot)$ is uniformly bounded on it)

$$\phi(-\Phi^{-1}(x) - m_0) \frac{t}{t+\varepsilon^\theta} \leq [CxL_C(x)] \frac{t}{t+\varepsilon^\theta} \leq Cx^{2\alpha}/L(x) = \frac{C}{w(x)^2}.$$

Hence Propositions 4.13 and 4.14 verify the WL-condition in Theorem 3.3 and Lemma 4.10 verifies the envelope function condition therein. Hence by part (i) of Proposition 3.5 and noting the distribution functions F_t of B_t are strictly increasing, we conclude the proof. \square

APPENDIX

In this appendix, we give the proof of Lemma 2.4.

Proof of Lemma 2.4. We denote the restricted process $\{G(t) : t \in T_0\}$ by G_0 . Then almost surely its sample paths are uniformly continuous on T_0 . Each sample path can be extended to a uniformly continuous sample path on T . Indeed, if we let $G_0(\omega)$ be a sample path and $t \in T$, then there is a sequence, say $(t_m) \subset T_0$, such that $d_G(t_m, t) \rightarrow 0$ as $m \rightarrow \infty$ and define $\tilde{G}(t)(\omega) := \lim_{m \rightarrow \infty} G(t_m)(\omega)$. It's easy to see it's well defined and is uniformly d_G continuous on T . Moreover, in view of its characteristic function, $\tilde{G}(t)$ is normal. Let $\tilde{\rho}$ be the covariance of \tilde{G} . It remains to show $\rho = \tilde{\rho}$. But that ρ and $\tilde{\rho}$ coincide on $T_0 \times T_0$ implies they coincide on $T \times T$. Indeed, for any $s, t \in T$, we can find a sequences (s_m) and (t_m) in T_0 , such that $d_G(s_m, s) \rightarrow 0$ and $d_G(t_m, t) \rightarrow 0$. Then $|\rho(s, t) - \rho(s_m, t_m)| \rightarrow 0$. \square

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